An Analytic Approach to Credit Risk of Large Corporate Bond and Loan Portfolios*

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Proof of Theorem ??: Along the lines of the previous proof, we have to consider

$$P(C > \pi^* - u_1) = P\left[\cup_{G\in\mathcal{G}}\left\{\sum_{j\in G}\lambda_j\hat{\pi}_j\Phi\left(\frac{s-|\hat{R}_j|\hat{v}_j^\top Y}{\sqrt{1-\hat{R}_j^2}}\right) > \pi^* - u_1\right\}\right]. \quad (1)$$

The first step is to prove that the events inside the square brackets are disjoint. To see this for $u_1 \downarrow 0$, let $G_1, G_2 \in \mathcal{G}$ with $G_1 \neq G_2$. Consider u_1 arbitrarily small and a region for Y such that for $j = 1, 2$,

$$\sum_{j\in G_i}\lambda_j\hat{\pi}_j\Phi\left(\frac{s-|\hat{R}_j|\hat{v}_j^\top Y}{\sqrt{1-\hat{R}_j^2}}\right) > \pi^* - u_1. \quad (2)$$

As there is no subset G_2^s of G_2 such that the inequality (2) is also satisfied for G_2^s, there must be a constant $k > 0$ such that

$$\sum_{j\in G_2\backslash G_1}\lambda_j\hat{\pi}_j\Phi\left(\frac{s-|\hat{R}_j|\hat{v}_j^\top Y}{\sqrt{1-\hat{R}_j^2}}\right) > k,$$

implying

$$\sum_{j\in G_2\cup G_1}\lambda_j\hat{\pi}_j\Phi\left(\frac{s-|\hat{R}_j|\hat{v}_j^\top Y}{\sqrt{1-\hat{R}_j^2}}\right) > \pi^* + k - u_1,$$

in the region for Y considered. This, however, contradicts the definition of π^*.

We now have for $u \downarrow 0$,

$$P(C > \pi^* - u_1) \stackrel{a}{=} \sum_{G\in\mathcal{G}} P\left[\sum_{j\in G}\lambda_j\hat{\pi}_j\Phi\left(\frac{s-|\hat{R}_j|\hat{v}_j^\top Y}{\sqrt{1-\hat{R}_j^2}}\right) > \pi^* - u_1\right]. \quad (3)$$

Define $a_j = s/\sqrt{1-\hat{R}_j^2}$ and $b_j = |\hat{R}_j|\hat{v}_j/\sqrt{1-R_j^2}$, and $\hat{\lambda}_j = \lambda_j\hat{\pi}_j$. Then the probabilities inside the sum in (3) simplify to

$$P\left[\sum_{j\in G}\hat{\lambda}_j\Phi\left(a_j - b_j^\top Y\right) > \pi^* - u_1\right]. \quad (4)$$

Now split Y in polar coordinates, $Y = R\theta$, with R^2 a χ^2_m variate, and θ uniform on a hyperglobe. The variates R and θ are independent. Now rewrite (4) as

$$\int P\left[\sum_{j\in G}\hat{\lambda}_j\Phi\left(a_j - Rb_j^\top\theta\right) > \pi^* - u_1\middle|\theta\right]P(d\theta). \quad (5)$$

Define $\bar{\Phi}(x) = 1 - \Phi(x)$. Then rewrite (5) as

$$\int P\left[\sum_{j\in G}\hat{\lambda}_j\bar{\Phi}\left(a_j - Rb_j^\top\theta\right) < u_1\middle|\theta\right]P(d\theta). \quad (6)$$

Now first consider the probabilities inside the integral. Define Θ as the set θ's for which $b_j^\top \theta < 0$ for all $j \in G$. Note that Θ constitutes the only set of θ's of interest. For other θ's, the probability inside the integral equals zero for $u_1 \downarrow 0$.

Next, make a subdivision of Θ into $\Theta_1, \ldots, \Theta_m$, such that we have $|b_j^\top \theta| < |b_i^\top \theta|$ for all $i \neq j$ and $\theta \in \Theta_j$. The Θ_j's are disjoint. Therefore, we can rewrite (6) as

$$\sum_{j \in G} \int_{\Theta_j} P\left[\hat{\lambda}_j \bar{\Phi}\left(a_j - R b_j^\top \theta\right) < u_1 \middle| \theta\right] P(d\theta). \tag{7}$$

Simplify the probability inside the integral as

$$P\left[R^2 > \left(\frac{\Phi^{-1}\left(u_1/\hat{\lambda}_j\right) + a_j}{b_j^\top \theta}\right)^2 \middle| \theta\right]. \tag{8}$$

From (6.5.4) and (6.5.32) in Abramowitz and Stegun (1970) we have

$$\int_x^\infty e^{-t} t^{a-1} dt = x^{a-1} e^{-x}(1 + O(x^{-1}))$$

for $x \to \infty$. Then from (26.4.19) from Abramowitz and Stegun it follows that for large x

$$P(R^2 > x^2) = \frac{(x/2)^{m/2-1} e^{-x^2/2}}{\Gamma(m/2)}(1 + O(x^{-2})).$$

We also have

$$\exp(-\Phi^{-1}(x)^2/2) \approx x \cdot L(x)$$

for $x \uparrow \infty$. Combining all these results and using the independence of R and θ, we can approximate (asymptotically) (8) by

$$\left(u_1/\hat{\lambda}_j\right)^{1/(b_j^\top \theta)^2}. \tag{9}$$

Again combining all results, we have for $u_1 \downarrow 0$

$$P(C > \pi^* - u_1) = \sum_{G \in \mathcal{G}} \sum_{j=1}^m \int_{\Theta_j} \left(u_1/\hat{\lambda}_j\right)^{1/(b_j^\top \theta)^2} P(d\theta). \tag{10}$$

As we are only interested in

$$\alpha = \lim_{u_1 \downarrow 0} \frac{\ln P(C > \pi^* - u_1)}{\ln u_1},$$

if follows from (10) that

$$\alpha = \min_{G \in \mathcal{G}} \min_{j \in G} \operatorname{ess} \inf_{\theta \in \Theta_j} (b_j^\top \theta)^{-2} = \min_{G \in \mathcal{G}} \min_{j \in G} \operatorname{ess} \inf_{\theta \in \Theta_j} \frac{1 - \hat{R}_j^2}{\hat{R}_j^2 (v_j^\top \theta)^2}, \tag{11}$$

where, to be precise, $\Theta_j = \Theta_j(G)$.

Remark: It is only a visual illusion that this result does not seem to nest the result for homogenous v_j. Indeed, there is a min over j rather than the max derived in the previous theorem. However, consider the case of homogenous v_j. In that case, we can simplify to a one-factor model by considering $v^\top Y$ instead of Y. Note that θ can only be 1 or -1 now. Using the proof of the present and the previous theorem, it is easy to see (focus for example on the case $m = 2$) that only one of the Θ_j's will be non-empty, and this non-empty Θ_j will contain either only 1 or only -1. The non-empty Θ_j is characterized by precisely that j for which $|b_j|$ is at its minimum, or $(1 - \hat{R}_j)^2/\hat{R}_j^2$ is at its maximum, see just above (7). So the minimum over j in (11) is correct, but one has to bear in mind that several of the $\Theta_j(G)$'s may be empty. We can easily accomodate this by defining the ess inf over an empty set to be $+\infty$.

Note that (11) can be simplified further. Define

$$\Theta^*(G) = \cup_{j \in G} \Theta_j(G),$$

then the minimum over j and the infimum over θ can be integrated. Note that conditional on a $\theta \in \Theta^*$, $j = j(\theta)$ is determined by the smallest $|b_j^\top \theta|$, i.e., by the maximum $(b_j^\top \theta)^{-2}$. Therefore, we have an equivalent expression for (11), namely

$$\alpha = \min_{G \in \mathcal{G}} \operatorname{ess} \inf_{\theta \in \Theta^*} \max_{j \in G} \frac{1 - \hat{R}_j^2}{\hat{R}_j^2 (v_j^\top \theta)^2}. \tag{12}$$

This completes the proof. ■