Monte Carlo Methods for Insurance Risk Computation

Ad Ridder (*Vrije University Amsterdam*)

Shaul Bar Lev (*University of Haifa*)

Rennes, July 3, 2018

The Problem

Compute the insurance risk,

$$\mathbb{P}\Big(\sum_{k=1}^{N} Y_k > x\Big),$$

where

- $Y_1, Y_2, \ldots \overset{\text{IID}}{\sim} F$ are individual insurance claims requested by policy holders of some specific insurance product during a certain period.
- N is the (random) counting variable of these claims.
- x is a large loss threshold.

History

This is a well-known and well-studied problem.

Traditionally, N is Poisson or negative binomial; Y is Gamma, lognormal, Pareto, ...

Or, more general, Y follows a Tweedie NEF (later more about this).

E.g., [Smyth and Jorgensen 2002,2005], [Dunn and Smith 2006,2008], [Vinogradov, Paris and Yanushkevichiene 2012, 2013].

In these studies, analytic or asymptotic expressions leading to approximate computations.

Our Contribution

We consider non-traditional distributions for which no numerical procedures exist.

We develop new sampling algorithms for these distributions.

For large loss levels we develop efficient importance sampling algorithms.

Our Model

The counting variable N follows a NEF (natural exponentials family) distribution with cubic variance function.

The claim size Y follows a reproducible NEF distribution.

Next slides more details on these issues and the distributions.

Motivation: data from car insurance claims support this modelling.

This talk is about modeling issues and constructing simulation algorithms.

It combines ideas from NEF (natural exponential families), reproducibility, Monte Carlo simulation, and importance sampling.

NEF

Definition

NEF is a family of parameterized probability distributions:

$$\mathcal{F} = \left\{ F_\theta : F_\theta(dx) = e^{\theta x - \kappa(\theta)}\, \mu(dx),\ \theta \in \Theta \right\}.$$

μ is a non-Dirac positive Radon measure on $\mathbb{R}$.

μ is called the kernel of the NEF.

$\kappa(\theta) = \log \int \exp(\theta x)\, \mu(dx)$ is cumulant function.

Mean and Variance Functions of NEF

Let X_θ the random variable associated with F_θ.

Definition

The mean function of the NEF is

$$m(\theta) = \mathbb{E}[X_\theta] = \kappa'(\theta).$$

The variance function of the NEF is

$$V(\theta) = \mathbb{V}ar[X_\theta] = \kappa''(\theta).$$

Parametrization by the Mean m

κ' is invertible, inverse denoted $\psi(m)$; i.e.

$$\kappa'(\theta) = m \quad \Leftrightarrow \quad \psi(m) = \theta.$$

Let $\psi_1(m) = \kappa\big(\psi(m)\big)$.

NEF by mean

$$\mathcal{F} = \Big\{ F_m : F_m(dx) = e^{\psi(m)x - \psi_1(m)}\, \mu(dx),\ m \in \mathcal{M} \Big\}.$$

Calculus

Denote the variance function by $V(m) = V\big(\psi(m)\big)$.

Easy to show that

$\psi(\cdot)$ is a primitive of $1/V(m)$; i.e.

$$\psi(m) = \int \frac{1}{V(m)}\, dm.$$

$\psi_1(\cdot)$ is a primitive of $m/V(m)$; i.e.

$$\psi_1(m) = \int \frac{m}{V(m)}\, dm.$$

Reverse NEF Modeling

Suppose the variance function $V(m)$ is given.

For instance, $V(m) = m$.

Our task is to compute the $\psi(m), \psi_1(m)$ functions, and the kernel $\mu(x)$.

Worked out in Letac&Mora (1990) and Bar-Lev&Kokonendji (2017).

Our Counting Variables

NEF counting variable $N \in \{0, 1, \ldots\}$.

Cubic variance function: $V(m) = O(m^3)$.

Examples [Letac and Mora 1990]:

- Abel: $V(m) = m\left(1 + \frac{m}{p}\right)^2$.
- Arcsine: $V(m) = m\left(1 + \frac{m^2}{p^2}\right)$.
- Takacs: $V(m) = m\left(1 + \frac{m}{p}\right)\left(1 + \frac{2m}{p}\right)$.

Note, all contain a dispersion parameter $p > 0$.

Analysis of these Counting Variables

For these three counting variables the kernel $\mu(n)$ is given in [Letac and Mora 1990].

We derived associated $\psi(m)$ and $\psi_1(m)$ functions.

In this way we obtain our (nontrivial) counting distributions.

Example: Abel NEF

Kernel:

$$\mu(n) = \frac{1}{n!} p(p+n)^{n-1}, \quad n = 0, 1, \ldots.$$

And

$$\psi(m) = \log \frac{m}{m+p} - \frac{m}{m+p};$$
$$\psi_1(m) = \frac{pm}{m+p}.$$

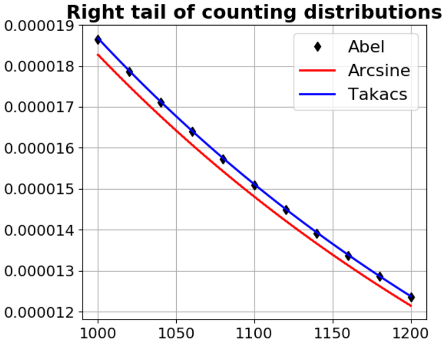
Thus

$$f(n) = \mathbb{P}(N = n) = \frac{1}{n!} \, \frac{p}{p+n} \left(\frac{m(p+n)}{m+p} \, e^{-\frac{m}{m+p}} \right)^n e^{-\frac{pm}{m+p}}$$

Generalized Poisson.

Large Tails

From data (case study), $m = 70$; $V = 52181$, we can fit these counting distributions.
Our counting distributions show fat tails.
The Poisson probabilities (only based on m) in this region are virtually zero.



Sampling from these Counting Distributions

We have developed accept-reject algorithms.

The algorithms use the same dominating proposal as the accept-reject for sampling from a Zipf distribution.

Derivations has two steps.

- First step: analyze asymptotics of the kernel $\mu(n), n \to \infty$;
- Second step: do the calculus for bounding $f(n)$.

Zipf Distribution

See Devroye (1986):

$$\frac{1}{\zeta(3/2)}\,\frac{1}{n\sqrt{n}}, \quad n = 1, 2, \ldots.$$

Sampling from this Zipf distribution is done by an accept-reject algorithm using the dominating pmf $b(n)$ of the random variable $\lfloor U^{-2} \rfloor$.

We have shown that our counting distributions are dominated as well by $b(n)$.

Acceptance ratios are 0.25 (Abel), 0.34 (Arcsine), and 0.23 (Takacs).

Our Claim Sizes

Reproducible NEF positive variable $Y > 0$.

Definition [Bar-Lev and Enis 1986]

Let $\mathcal{F}$ be a NEF parameterized by θ:

$$\mathcal{F} = \left\{ F_\theta : F_\theta(dx) = e^{\theta x - \kappa(\theta)}\, \mu(dx),\ \theta \in \Theta \right\}.$$

Let $Y_1, Y_2, \ldots \overset{\text{IID}}{\sim} F_\theta$. Denote $S_n = \sum_{i=1}^n Y_i$.

The NEF is reproducible if there exist a sequence of real numbers $(c_n)_{n \geq 1}$, and a sequence of mappings $\{g_n : \Theta \to \Theta, n \geq 1\}$, such that for all $n \in \mathbb{N}$ and for all $\theta \in \Theta$,

$$c_n S_n \overset{\mathcal{D}}{\sim} F_{g_n(\theta)} \in \mathcal{F}.$$

Theorem [Bar-Lev and Casalis 2003]

NEF is reproducible iff it possesses a power variance function.

Examples of Reproducible NEF's

- Gamma: $V(m) = \frac{m^2}{p}$.
- Inverse Gaussian: $V(m) = pm^3$.
- Positive α-stable: $V(m) = \alpha(1-\alpha)\frac{m^p}{\alpha^p}$, where $\alpha \in (0,1)$, $p = (2-\alpha)/(1-\alpha) > 2$.

Sampling algorithms are available.

Insurance Risk Computation

Recall to compute $\ell(x) = \mathbb{P}(S_N > x)$.

Monte Carlo simulation is now easy:

- Repeat
  1. Generate N.
  2. Generate S_N using reproducibility.
  3. Check $S_N > x$.
- Until sufficiently many observations.

Criterion: standard error of estimator $\approx$ 10% of the computed estimate.

M is the required sample size.

Numerical Experiment

Data from car insurance shows empirical mean and variance for the count variable N and the claim size Y, respectively

$$\hat{m}_N = 70.6;\ \hat{v}_N = 52181.52; \quad \hat{m}_Y = 4.66;\ \hat{v}_Y = 265.24.$$

From these the distribution parameters are obtained by fitting first two moments.

Model: Abel counting and inverse Gaussian claims.

x	M	$\widehat{\ell}$	std. error
5000	9000	1.08e-02	1.09e-03
10000	37000	2.59e-03	2.64e-04
15000	150000	6.47e-04	6.56e-05
20000	410000	2.37e-04	2.40e-05
25000	1020000	9.51e-05	9.66e-06

M grows exponentially in x.

Importance Sampling

Simulate the model with the same distributions but with other parameters.

Say $\widetilde{N}$ and $\widetilde{Y}$ are the corresponding variables.

Make the event $\{\widetilde{S}_{\tilde{N}} > x\}$ more often to happen.

For instance by requiring $\mathbb{E}[\widetilde{S}_{\tilde{N}}] = \mathbb{E}[\widetilde{N}] \times \mathbb{E}[\widetilde{Y}] = x$.

For unbiasedness you need the likelihood ratio

$$W(n, s) = \frac{\mathbb{P}(N = n)}{\mathbb{P}(\widetilde{N} = n)} \times \frac{f_S(s)}{f_{\widetilde{S}}(s)}.$$

Choosing the Change of Measure

Apply an exponential change of measure.

Find tilting parameter η such that

$$\mathbb{E}[\widetilde{S}_{\tilde{N}}] = \mathbb{E}_\eta[S_N] = x$$

for getting efficient importance sampling.

Since $\mathbb{E}[\widetilde{S}_{\tilde{N}}] = \mathbb{E}[\widetilde{Y}] \times \mathbb{E}[\widetilde{N}]$, find η s.t.

$$\mathbb{E}_\eta[Y] \times \mathbb{E}_\eta[N] = x.$$

Numerical Experiment (revisited)

Same data as above.

Same model: Abel counting and inverse Gaussian claims.

x	M	$\widehat{\ell}$	std. error
5000	4000	1.01e-02	9.09e-04
10000	6000	2.46e-03	2.43e-04
15000	10000	7.18e-04	6.88e-05
20000	14000	2.22e-04	2.23e-05
25000	16000	8.48e-05	8.40e-06
30000	20000	3.59e-05	3.65e-06
35000	26000	1.29e-05	1.25e-06
40000	34000	4.42e-06	4.41e-07
45000	34000	2.18e-06	2.16e-07
50000	40000	7.68e-07	7.78e-08

M grows linearly in x.

Case Study

Arcsine counting variable and positive α-stable claim sizes gave best fit.

Simulation results:

	Monte Carlo			importance sampling		
x	M	$\widehat{\ell}$	std. error	M	$\widehat{\ell}$	std. error
5000	10000	1.02e-02	1.00e-03	4000	9.89e-03	9.39e-04
10000	46000	2.11e-03	2.14e-04	7000	2.23e-03	2.23e-04
15000	128000	7.64e-04	7.73e-05	9000	8.12e-04	8.20e-05
20000	394000	2.49e-04	2.51e-05	14000	2.21e-04	2.25e-05
25000	1360000	7.13e-05	7.24e-06	16000	8.37e-05	8.52e-06
30000				20000	4.18e-05	4.13e-06
35000				24000	1.42e-05	1.42e-06
40000				30000	5.10e-06	5.15e-07
45000				36000	2.31e-06	2.35e-07
50000				38000	1.08e-06	1.08e-07

Same observation: importance sampling is efficient.

Work-Normalized Variances

Variance of the estimator multiplied with the total simulation time.

x	MC	IS
5000	5.54e-05	4.52e-06
10000	1.49e-05	1.69e-07
15000	7.71e-06	1.80e-07
20000	2.72e-06	2.26e-08
25000	7.84e-07	5.21e-09

Conclusion

Insurance risk computations.
Probabilistic model involving non-trivial random variables.

NEF modeling is a flexible technique for handling empirical mean and variance to fit models.

Monte Carlo simulations made more efficient by

- Exploiting reproducibility.
- Importance sampling by an exponential change of measure.

Thank you for your attention.