On (r, Q) Inventory Systems with Truncated Normal Lead Times and Poisson Demands

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Abstract: In this note we shall deal with a continuous review (r, Q) inventory control system. Demands for a single product item occur at epochs generated by a Poisson process, and the replenishment lead time has a truncated normal distribution. We shall derive expressions for the demand probabilities during lead times based on exact expressions for 'tail moments' of the standard normal distribution. The program for finding an optimal reorder point r and an optimal order quantity Q under service level constraints is solved numerically.

1 Introduction

Consider a continuous review (r, Q) inventory control system with stochastic lead times and demands. More precisely, we assume that demands occur at epochs generated by a Poisson process with rate λ and that each demand size equals one unit. The lead time L is stochastic with density

$$f_L(t) = \begin{cases} \frac{1}{\Phi(\mu/\sigma)} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(t-\mu)^2} & t > 0 \\ 0 & t \leq 0 \end{cases} \tag{1}$$

(with $\mu > 0$ and $\sigma \neq 0$) where $\Phi(\cdot)$ stands for the standard normal cumulative distribution function. It says that L is truncated normally distributed. Mean μ_L and variance σ_L^2 of L are expressed using tail moments of the standard normal density function ϕ,

$$M_i(x) = \int_x^\infty u^i \phi(u) du \tag{2}$$

for $x \in \mathbb{R}$ and $i = 0, 1, \ldots$. In section 2 we shall derive expressions for these moments, here we use the result for $i \leq 2$:

$$M_0(x) = \Phi(-x),\ M_1(x) = \phi(x),\ M_2(x) = x\phi(x) + M_0(x)$$

Then

$$\mu_L = \mu + \sigma \frac{M_1(-\mu/\sigma)}{\Phi(\mu/\sigma)}, \qquad \sigma_L^2 = \sigma^2 \left(\frac{M_2(-\mu/\sigma)}{\Phi(\mu/\sigma)} - \frac{M_1^2(-\mu/\sigma)}{\Phi^2(\mu/\sigma)} \right)$$

Let the stochastic variable W denote the total demand during lead time. Its distribution is a main issue in modelling inventory systems since it has impact on safety stocks, decision variables and performance. Commonly one approximates it by a normal distribution for getting 'easy' expressions [3]. Also weibull distributions have been used as approximations that are numerical tractable [4]. In some cases it is possible to derive the exact distributions, see [2] for an overview. The particular model of this paper yields exact expressions by using (1) and (2) and executing some algebra:

$$p_W(i) = P(W = i) = \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^i}{i!} f_L(t) dt = G \frac{(\lambda\sigma)^i}{i!} \sum_{j=0}^{i} \binom{0}{i} j \left(\frac{\hat{\mu}}{\sigma}\right)^j M_j(-\frac{\hat{\mu}}{\sigma}) \tag{3}$$

where

$$G = \frac{e^{-\lambda\mu + \frac{1}{2}\lambda^2\sigma^2}}{\Phi(\frac{\mu}{\sigma})} \text{ and } \hat{\mu} = \mu - \lambda\sigma^2$$

Note that the mean and variance of W are

$$\mu_W = \sum_{i=0}^{\infty} i p_W(i) = \lambda\mu_L \text{ and } \sigma_W^2 = \lambda^2\sigma_L^2 + \mu_W$$

The probabilities (3) are numerically tractable only for 'slow moving' items, i.e. $\lambda \leq \mu/\sigma^2$. In [1] these probabilities are approximated by relaxing the truncation of the lead time distribution:

$$\tilde{p}_W(i) = \int_{-\infty}^{\infty} e^{-\lambda t} \frac{(\lambda t)^i}{i!} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(t-\mu)^2} dt \tag{4}$$

For slow moving items (4) is the density of a Hermite distribution [1]. Numerical experiments show that for (absolute) small σ and for relative large λ (compared to μ) the exact and approximate values agree with 4 decimales. As for 'fast moving' items where we cannot calculate the exact probabilities or use the hermite approximation, we either use an approximation of the lead time demand (e.g. normal distribution), or we approximate the demand and/or lead time distribution in a way that a well known model arises [2].

2 Expressions for $M_i(x)$

Theorem 1 of this section expresses the tail moments $M_i(x)$ in terms of values of the standard normal distribution and density. First we introduce two collections of integers.

Definition 1 *We introduce for $n = 0, 1, \ldots$ and $k = 0, 1, \ldots, n$ the integers $a_k^{(n)}$ by*

$$\begin{aligned} a_0^{(0)} &= 1 \\ a_k^{(n)} &= 2na_k^{(n-1)}, \quad n = 1, 2, \ldots, \; k = 0, 1, \ldots n-1 \\ a_n^{(n)} &= 1, \quad n = 1, 2, \ldots \end{aligned}$$

and the integers $b_k^{(n)}$ *by*

$$\begin{aligned} b_0^{(0)} &= 1 \\ b_k^{(n)} &= (2n+1)b_k^{(n-1)}, \quad n = 1,2,\dots,\ k = 0,1,\dots n-1 \\ b_n^{(n)} &= 1, \quad n = 1,2,\dots \end{aligned}$$

Then it is easy to show by induction on n the following relations of these integers.

Lemma 1

$$\begin{aligned} a_k^{(n)} &= (2k+2)a_{k+1}^{(n)}, \quad n \geq 1,\ k \leq n-1 \\ b_k^{(n)} &= (2k+3)b_{k+1}^{(n)}, \quad n \geq 1,\ k \leq n-1 \\ b_k^{(n)} &= a_k^{(n)} + \sum_{\ell=k+1}^{n} a_\ell^{(n)} b_k^{(\ell-1)}, \quad n \geq 1,\ k \leq n \end{aligned}$$

The main result of this section is the following statement.

Theorem 1

$$M_i(x) = \alpha_i \Phi(-x) + \beta_i(x)\phi(x)$$

with

(i) for $i = 0$

$$\alpha_0 = 1,\ \beta_0(x) = 0$$

(ii) for $i = 2n+1$ *odd* $(n \geq 0)$

$$\alpha_i = 0,\ \beta_i(x) = \sum_{k=0}^{n} a_k^{(n)} x^{2k}$$

(iii) for $i = 2n$ *even* $(n \geq 1)$

$$\alpha_i = b_0^{(n-1)},\ \beta_i(x) = \sum_{k=0}^{n-1} b_k^{(n-1)} x^{2k+1}$$

PROOF: (i) is clear.

(ii) For $n=0$ the statement says $M_1(x)=\phi(x)$ which is an easy exercise. Suppose that (ii) holds for $n \le m$ and set $n=m+1$ $(i=2m+3)$. Then by using a change of variable, integration by parts, the induction hypothesis and Lemma 1,

$$
\begin{aligned}
M_{2m+3}(x) = & \int_x^\infty u^{2m+3}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}du = \int_{x^2}^\infty v^{m+1}\frac{1}{2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}v}dv \\
= \quad & x^{2m+2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2} + (m+1)\int_{x^2}^\infty v^m\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}v}dv \\
= \quad & x^{2m+2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2} + 2(m+1)\int_x^\infty u^{2m+1}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}du \\
= \quad & x^{2m+2}\phi(x) + 2(m+1)M_{2m+1}(x) = x^{2(m+1)}\phi(x) + 2(m+1)\sum_{k=0}^{m}a_k^{(m)}x^{2k}\phi(x) \\
= \quad & \sum_{k=0}^{m+1}a_k^{(m+1)}x^{2k}\phi(x)
\end{aligned}
$$

(iii) First show by induction on n (and using part (ii)) that

$$
M_i(x) = \sum_{k=0}^{n-1}a_k^{(n-1)}\left\{x^{2k+1}\phi(x) + M_{2k}(x)\right\},\ i=2n, n\ge 1 \qquad (5)
$$

Then when $n=1$ $(i=2)$ (5) says

$$
M_2(x) = x\phi(x) + M_0(x) = x\phi(x) + \Phi(-x)
$$

This gives (iii) because $\alpha_2 = b_0^{(0)} = 1$ and $\beta_2(x) = b_0^{(0)}x = x$. Suppose the statement holds for $n \le m$ and set $n=m+1$ $(i=2m+2)$, then by using (5), the induction hypothesis and Lemma 1,

$$
\begin{aligned}
M_{2m+2}(x) &= \sum_{k=0}^{m} a_k^{(m)} \left\{ x^{2k+1}\phi(x) + M_{2k}(x) \right\} \\
&= \sum_{k=0}^{m} a_k^{(m)} x^{2k+1}\phi(x) + a_0^{(m)}\Phi(-x) + \sum_{k=1}^{m} a_k^{(m)} \left\{ b_0^{(k-1)}\Phi(-x) + \sum_{\ell=0}^{k-1} b_\ell^{(k-1)} x^{2\ell+1}\phi(x) \right\} \\
&= \left\{ a_0^{(m)} + \sum_{k=1}^{m} a_k^{(m)} b_0^{(k-1)} \right\} \Phi(-x) + \sum_{k=0}^{m} a_k^{(m)} x^{2k+1}\phi(x) + \sum_{k=1}^{m}\sum_{\ell=0}^{k-1} a_k^{(m)} b_\ell^{(k-1)} x^{2\ell+1}\phi(x) \\
&= b_0^{(m)}\Phi(-x) + \sum_{k=0}^{m} a_k^{(m)} x^{2k+1}\phi(x) + \sum_{\ell=0}^{m-1}\sum_{k=\ell+1}^{m} a_k^{(m)} b_\ell^{(k-1)} x^{2\ell+1}\phi(x) \\
&= b_0^{(m)}\Phi(-x) + \sum_{k=0}^{m-1} \left\{ a_k^{(m)} + \sum_{\ell=k+1}^{m} a_\ell^{(m)} b_k^{(\ell-1)} \right\} x^{2k+1}\phi(x) + x^{2m+1}\phi(x) \\
&= b_0^{(m)}\Phi(-x) + \sum_{k=0}^{m} b_k^{(m)} x^{2k+1}\phi(x)
\end{aligned}
$$

□

3 Optimization programs

In this section we use the lead time demand probabilities (3) in the average cost function per unit time of the (r, Q) inventory system with backordering. Particularly we are interested in the effect of the lead time variability on the decision variables r and Q. Also we compare the results with those obtained applying the approximation (4).

Assuming no backordering costs we consider minimizing the average costs per unit time under service level constraint. The cost function becomes under the usual assumptions [3].

$$\mathcal{K}(r, Q) = A\frac{\lambda}{Q} + IC\left(\frac{Q+1}{2} + r - \mu_W\right)$$

where A is the cost of placing an order, I the inventory carrying charge, C unit cost of the item

We shall consider two optimization programs.

(I) minimize the cost function under service level constraint on the probability of no stockout during a replenishment cycle,

$$B_1(r) = P(W \leq r) = \sum_{i=0}^{r} p_W(i) \geq 1 - \epsilon_1$$

(II) minimize the cost function under service level constraint on the fraction of demand satisfied directly from on-hand inventory,

$$B_2(r, Q) = 1 - \frac{1}{Q} \sum_{i=r}^{\infty} (i - r) p_W(i) \geq 1 - \epsilon_2$$

The ϵ's are typically of the order of 5%. The programs are solved by calculating the differentials

$$\Delta_r \mathcal{K}(r, Q) = \mathcal{K}(r+1, Q) - \mathcal{K}(r, Q) \text{ and } \Delta_Q \mathcal{K}(r, Q) = \mathcal{K}(r, Q+1) - \mathcal{K}(r, Q)$$

Then we use structural properties of these differentials and apply standard numerical procedures to find the optimal reorder point r^* and optimal order size Q^*. The following tables summarize some numerical experiments for slow moving items, keeping λ and μ fixed and varying the lead time variance via σ. The cost factors are $A = 500, IC = 25$, the ϵ's are 5%.

	Program I		Program II	
σ	r^*	Q^*	r^*	Q^*
0.05	8	6	5	9
0.5	8	6	5	9
0.75	8	6	6	9
1.25	8	6	6	9
1.5	9	6	6	9
1.75	9	6	6	9
1.95	9	6	7	9

$\lambda = 1, \mu = 4, 0 < \sigma < 2$

	Program I		Program II	
σ	r^*	Q^*	r^*	Q^*
0.05	5	6	3	8
0.25	5	6	3	8
0.5	5	6	3	8
1.0	5	6	3	8
1.25	6	6	3	9
1.4	6	6	4	9

$\lambda = 1, \mu = 2, 0 < \sigma < 1.414$

σ	Program I r^*	Q^*	Program II r^*	Q^*
0.05	6	8	4	10
0.5	6	8	4	10
0.75	7	8	4	10
1.0	7	8	4	11
1.15	8	8	5	10

$\lambda = 1.5, \mu = 2, 0 < \sigma < 1.155$

σ	Program I r^*	Q^*	Program II r^*	Q^*
0.05	8	9	5	11
0.6	8	9	5	11
0.7	8	9	5	12
0.95	9	9	6	12

$\lambda = 2, \mu = 2, 0 < \sigma < 1.0$

From these results we conclude that the optimal reorder point and optimal order size are (almost) insensitive for σ (and hence for variability in lead time). We performed the same experiments for the hermites approximation of the lead time demand. It turns out that we find (in most cases) the same numbers. The approximation works so well because we need rounding to integers.

References

[1] U. Bagchi, J.C. Hayya and J.K. Ord. The Hermite Distribution as a Model of Demand during Lead Time for Slow-Moving Items. *Decision Sciences*, 14:447 – 466, 1983.

[2] U. Bagchi, J.C. Hayya and J.K. Ord. Modeling Demand during Lead Time. *Decision Sciences*, 15:157 – 176, 1984.

[3] E.A. Silver and R. Peterson. *Decision Systems for Inventory Management and Production Planning*. John Wiley, New York, second edition, 1985.

[4] P.R. Tadikamalla. Applications of the Weibull Distribution in Inventory Control. *Journal of the Operational Research Society*, 29:77 – 83, 1978.