A generalization of the classical secretary problem

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1 Introduction

The classical secretary problem is a well known optimal stopping problem from probability theory. It is usually described by different real life examples, notably the process of hiring a secretary. Imagine a company manager in need of a secretary. Our manager wants to hire only the best secretary from a given set of n candidates, where n is known. No candidate is equally as qualified as another. The manager decides to interview the candidates one by one in a random fashion. Every time he has interviewed a candidate he has to decide immediately whether to hire her or to reject her and interview the next one. During the interview process he can only judge the qualities of those candidates he has already interviewed. This means that for every candidate he has observed, there might be an even better qualified one within the set of candidates yet to be observed. Of course the idea is that by the time only a small number of candidates remain unobserved, a recently interviewed candidate that is relatively best will probably also be the overall best candidate.

There is abundant research literature on this classical secretary problem, for which we refer to Ferguson [2] for an historical note and an extensive bibliography. The exact optimal policy is known, and may be derived by various methods, see for instance Dynkin and Yushkevich [1], and Gilbert and Mosteller [4]. Also, many variations and generalizations of the original problem have been introduced and analysed. One of these generalizations is the focus of our paper, namely the problem to select one of the b best, where $1 \leq b \leq n$ is some preassigned number (notice that $b = 1$ is the classical secretary problem). Originally, this problem was introduced by Gusein-Zade [5], who derived the structure of the optimal policy: there is a sequence $0 \leq s_1 < s_2 < \cdots < s_b \leq s_{b+1} = n-1$ of position thresholds such that when candidate i is presented, and judged to have relative rank k among the first i candidates[1], then the optimal

[1] It is most convenient to rank the candidates $1, 2, \ldots, n$, with rank 1 being the best, rank 2 being second best, etc.

decision says

$$
\begin{aligned}
&i \le s_1: \text{ continue whatever } k \text{ is;}\\
&s_j + 1 \le i \le s_{j+1} \ (\text{where } j = 1,\ldots,b): \begin{cases} \text{stop if } k \le j \\ \text{continue if } k > j; \end{cases}\\
&i = n: \text{ stop whatever } k \text{ is.}
\end{aligned}
$$

Furthermore, [5] gave an algorithm to compute these thresholds, and derived asymptotic expressions (as $n \to \infty$) for the $b = 2$ case. Also Frank and Samuels [3] proposed an algorithm, and gave the limiting (as $n \to \infty$) probabilities and limiting proportional thresholds s_j/n.

The algorithms of [3, 5] are based on dynamic programming, which means that the optimal thresholds s_j, and the optimal winning probability are determined numerically. The next interest was to find analytic expressions. To our best knowledge, this has been resolved only for $b = 2$ by Gilbert and Mosteller [4], and for $b = 3$ by Quine and Law [6]. Although the latter claim that their approach is applicable to produce exact results for any b, it is clear that the expressions become rather untractable for larger b. This has inspired us to develop approximate results for larger b.

We consider two approximate policies for the general b case: single-level policies, and double-level policies. A single-level policy is given by a single position threshold s in conjunction with a rank level r, such that when candidate i is presented, and judged to have relative rank k among the first i candidates, then the policy says

$$
\begin{aligned}
&i \le s: \text{ continue whatever } k \text{ is;}\\
&s + 1 \le i \le n - 1 \ : \begin{cases} \text{stop if } k \le r \\ \text{continue if } k > r; \end{cases}\\
&i = n: \text{ stop whatever } k \text{ is.}
\end{aligned}
$$

A double-level policy is given by two position thresholds $s_1 < s_2$ in conjunction with two rank levels $r_1 < r_2$, such that when candidate i is presented, and judged to have relative rank k among the first i candidates, then the policy says

$$
\begin{aligned}
&i \le s_1: \text{ continue whatever } k \text{ is;}\\
&s_1 + 1 \le i \le s_2 \ : \begin{cases} \text{stop if } k \le r_1 \\ \text{continue if } k > r_1; \end{cases}\\
&s_2 + 1 \le i \le n - 1 \ : \begin{cases} \text{stop if } k \le r_2 \\ \text{continue if } k > r_2; \end{cases}\\
&i = n: \text{ stop whatever } k \text{ is.}
\end{aligned}
$$

We shall derive the exact winning probability for these two approximate policies, when the threshold and level parameters are given. These expressions can then used easily to compute the optimal single-level and the optimal double-level policies, i.e., we optimize the winning probabilities (under these level policies) with respect to their threshold and level parameters. The most important result is that the winning probabilities of the optimal double-level policies are extremely close to the winning probabilities of the optimal policies (with the b thresholds), specifically for larger b, see Table 1 in Section 4. In other words, we have found explicit formulas that approximate closely the winning probabilities for this generalized secretary problem.

2 Single-level policies

Before we consider the single-level policies we first introduce some notation we use throughout this paper. The absolute rank of the i-th object is denoted by X_i, while the relative rank of the i-th object is denoted by Y_i. Ranks run from 1 to n, and we say that rank i is higher than rank j when $i < j$. Moreover for natural numbers x and n, the falling factorial $x(x-1)\dots(x-n+1)$ is denoted by $(x)_n$.

Single-level policies are determined by two integer parameters: s (called the position threshold) and r (called the rank level). Following such a single-level policy objects are considered to be selected from position $s+1$ and then the first one encountered with a relative rank higher or equal than r is picked. Moreover, we assume that if the first $n-1$ items are not picked that then the last object is certainly picked independent of its relative rank Y_n. Let $\pi = \pi(s,r)$ be such a policy with $1 \le r \le b$ and $r \le s \le n-1$; we discard the trivial cases of $s = n$ (never stop before the last object), and $s < r$ (stop at position $s+1$), and denote the probability of success by $P_{\text{SLP}}(\pi)$. Thus $P_{\text{SLP}}(\pi)$ is the probability that an object is picked with absolute rank higher than or equal to b if policy π is applied. Note: when we wish to express explicitly parameters (n,b,s,r) we denote it, otherwise we omit it.

Theorem 1. *For $r = 1,2,\dots,b$, and $r \le s \le n-1$:*

$$P_{\text{SLP}}(\pi(s,r)) = \sum_{i=s+1}^{n-1} \frac{(s)_r}{(i-1)_r} \left(\frac{r}{n} + \frac{1}{n} \sum_{j=r+1}^{b} \sum_{k=1}^{r} \frac{\binom{j-1}{k-1}\binom{n-j}{i-k}}{\binom{n-1}{i-1}} \right) + \frac{(s)_r}{(n-1)_r}\,\frac{b}{n}.$$

Before proving this expression we need two auxiliary results (we give no proofs!).

Lemma 2. *For $s = r,\dots,n-2$ and $i = s+2, s+3,\dots,n$ we have that*

$$\mathbb{P}(\min\{Y_{s+1}, Y_{s+2},\dots,Y_{i-1}\} > r) = \frac{(s)_r}{(i-1)_r}$$

Lemma 3. *For $i = s+1, s+2,\dots,n-1$ and $r = 1,2,\dots,b$ we have that*

$$\mathbb{P}(Y_i \le r | X_i = j) = \begin{cases} 1 & \text{for } j = 1,2,\dots,r \\ \sum_{k=1}^{r} \frac{\binom{j-1}{k-1}\binom{n-j}{i-k}}{\binom{n-1}{i-1}} & \text{for } j = r+1, r+2,\dots,b. \end{cases}$$

Proof. (Of Theorem 1) The case $s = n-1$ is trivial because then $P_{\text{SLP}}(\pi(n-1,r)) = \mathbb{P}(X_n \leq b) = \frac{b}{n}$. Let $r \leq s \leq n-2$. For $i = s+1, s+2, \dots, n$ and $j = 1, 2, \dots, b$ let A^i_j be the event that $X_i = j$ and policy $\pi(s,r)$ picks the object at position i:

$$A^i_j = \{\min\{Y_{s+1}, Y_{s+2}, \dots, Y_{i-1}\} > r, Y_i \leq r, X_i = j\}.$$

Thus,

$$\begin{aligned}
P_{\text{SLP}}(\pi(s,r)) &= \sum_{i=s+1}^{n} \sum_{j=1}^{b} \mathbb{P}(A^i_j) \\
&= \sum_{j=1}^{b} \mathbb{P}(A^{s+1}_j) + \sum_{i=s+2}^{n-1} \sum_{j=1}^{b} \mathbb{P}(A^i_j) + \sum_{j=1}^{b} \mathbb{P}(A^n_j).
\end{aligned}$$

Cases $i = s+1$ and $i = n$ are treated separately. Notice that for $k < i$ the relative ranks Y_k are independent of both X_i and Y_i, thus (for $s+2 \leq i \leq n-1$)

$$\begin{aligned}
\mathbb{P}(A^i_j) &= \mathbb{P}(\min\{Y_{s+1}, Y_{s+2}, \dots, Y_{i-1}\} > r, Y_i \leq r, X_i = j) \\
&= \mathbb{P}(\min\{Y_{s+1}, Y_{s+2}, \dots, Y_{i-1}\} > r)\, \mathbb{P}(Y_i \leq r, X_i = j) \\
&= \mathbb{P}(\min\{Y_{s+1}, Y_{s+2}, \dots, Y_{i-1}\} > r)\, \mathbb{P}(Y_i \leq r | X_i = j)\, \mathbb{P}(X_i = j),
\end{aligned}$$

with $\mathbb{P}(X_i = j) = \frac{1}{n}$, and the other two factors were determined in Lemma 2 and Lemma 3. For $i = s+1$:

$$\mathbb{P}(A^{s+1}_j) = \mathbb{P}(X_{s+1} = j, Y_{s+1} \leq r) = \mathbb{P}(Y_{s+1} \leq r | X_{s+1} = j)\, \mathbb{P}(X_{s+1} = j),$$

and then apply Lemma 3 while noticing that $(s)_r/(i-1)_r = 1$. For $i = n$:

$$\begin{aligned}
\mathbb{P}(A^n_j) &= \mathbb{P}(\min\{Y_{s+1}, Y_{s+2}, \dots, Y_{n-1}\} > r, X_n = j) \\
&= \mathbb{P}(\min\{Y_{s+1}, Y_{s+2}, \dots, Y_{n-1}\} > r)\mathbb{P}(X_n = j),
\end{aligned}$$

and apply Lemma 2. □

We defer the comparison of the performance of single-level policies with the optimal policy to Section 4.

3 Double-level policies

A natural extension of the single-level policies is the class of double-level policies for the secretary problem where the objective is to pick one of the b best objects from n objects consecutively arriving one by one in the usual random fashion. Let be given two rank levels $1 \leq r_1 < r_2 \leq b$, and two position thresholds $r_1 \leq s_1 < s_2 \leq n-1$ (we discard the trivial cases of $s_2 = n$ which gives again a single-level policy, and $s_1 < r_1$ which leads to stopping at position $s_1 + 1$). The double-level policy says to observe the first s_1 presented objects without picking any; next, from objects at positions s_1+1 up to s_2 the first one encountered with a relative rank higher or equal than r_1 is picked; if no such object appears, the first object at positions s_2+1 up to $n-1$ is selected which has a relative rank of at least r_2; finally, if all these $n-1$ items are not picked, the last object is certainly picked independent of its relative rank Y_n. Slightly abusing, we denote again by $\pi=\pi(s,r)$ such a double-level policy and by $P_{\text{DLP}}(\pi)$ its winning probability. Similar to the proof of Theorem 1 we derive the winning probability.

Theorem 4. *The double-level policy given by rank levels $1\le r_1<r_2\le b$, and position thresholds $r_1\le s_1<s_2\le n-1$ has winning probability*

$$P_{\text{DLP}}(\pi(s,r)) = \sum_{i=s_1+1}^{s_2} \frac{(s_1)_{r_1}}{(i-1)_{r_1}} \left(\frac{r_1}{n} + \frac{1}{n} \sum_{j=r_1+1}^{b} \sum_{k=1}^{r_1} \frac{\binom{j-1}{k-1}\binom{n-j}{i-k}}{\binom{n-1}{i-1}} \right)$$
$$+ \sum_{i=s_2+1}^{n-1} \frac{(s_1)_{r_1}(s_2-r_1)_{r_2-r_1}}{(i-1)_{r_2}} \left(\frac{r_2}{n} + \frac{1}{n} \sum_{j=r_2+1}^{b} \sum_{k=1}^{r_2} \frac{\binom{j-1}{k-1}\binom{n-j}{i-k}}{\binom{n-1}{i-1}} \right)$$
$$+ \frac{(s_1)_{r_1}(s_2-r_1)_{r_2-r_1}}{(n-1)_{r_2}} \frac{b}{n}.$$

4 Numerical Results

We can find numerically the optimal single-level policy for a given number of candidates n, and a given worst allowable rank b, in a two-step approach as:

$$\max_{r=1,\dots,b} \max_{s=r,\dots,n-1} P_{\text{SLP}}(\pi(s,r)).$$

Thus, in the first step, we fix also a rank level r (between 1 and b). The function $\{r,\dots,n-1\}\to P_{\text{SLP}}(\pi(\cdot,r))$ is unimodal concave (this follows after a marginal analysis), and thus we can solve numerically for the optimal position threshold $s^*=s^*(r)$, and the associated winning probability $P_{\text{SLP}}(\pi(s^*,r))$. The second step is simply a complete enumeration to determine

$$\max\{P_{\text{SLP}}(\pi(s^*,r)) : r=1,\dots b\}.$$

However, it can be shown that the function $\{1,\dots,b\}\to P_{\text{SLP}}(\pi(s^*(\cdot),\cdot))$ is unimodal, which yields a shortcut in the second step. To check our numerical results, we have constructed an alternative method to find the optimal position threshold $s^*(r)$, given n,b,r, namely by dynamic programming.

Similarly, in the case of double-level policies, we have constructed a two-step approach, where the first step finds the optimal position thresholds $s_1^*=s_1^*(r_1,r_2)$ and $s_2^*=s_2^*(r_1,r_2)$ for any given pair of rank levels (r_1,r_2), and its associated winning probability $P_{\text{DLP}}(\pi(s^*,r))$ (vector notation for s and r). Then a straightforward search procedure determines

$$\max_{r_1=1,\dots,b-1} \max_{r_2=r_1+1,\dots,b} P_{\text{DLP}}(\pi(s^*,r)).$$

Finally, as mentioned in the introductory section, dynamic programming can be applied easily to obtain the optimal (multi-level) policy [3, 5]. We have implemented the algorithms for the optimal multi-level and optimal double-level policies on the website

`http://staff.feweb.vu.nl/aridder/java/best.html`

Table 1 gives the relative errors of the winning probabilities of the optimal single and double-level policies for $n = 100, 250$, and $n = 1000$, and for $b = 5, 10, \ldots, 25$, relatively to the corresponding optimal multi-level policies. The double-level policy gives extremely small errors for larger b, up to very large population sizes n. Also we notice that the errors (for a given b) increase slightly as n increases.

Table 1: *Relative errors (%) of the optimal single- and double-level policies.*

	single-level			double-level		
	$n = 100$	$n = 250$	$n = 1000$	$n = 100$	$n = 250$	$n = 1000$
$b = 5$	10.630	10.854	10.965	3.286	3.331	3.354
$b = 10$	5.262	5.674	5.876	1.702	1.841	1.911
$b = 15$	2.095	2.467	2.658	0.568	0.686	0.746
$b = 20$	0.739	0.996	1.131	0.155	0.221	0.258
$b = 25$	0.239	0.381	0.464	0.036	0.066	0.084

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